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Contiguous relation between the C-G coefficients of SU (n)

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A method for deriving the contiguous relations between the C–G coefficients of $SU(n)$, with different representation labels, has been discussed For $SU(2)$ the existence of only one such independent relation has been established In case of $SU(3)$ the contiguous relations involve the product of C–G coefficients, summed over the degenerate representations, which act as the building blocks in forming the orthonormalised C–G coefficients

1. Introduction

The successes of $SU(3)$, $SU(4)$ and $SU(6)$ in the past decade have stimulated among the physicists an interest about the unitary unimodular group in n-dimensions As a consequence, a number of works dealing with the various properties of the group and its representations have appeared in literature One of the important questions in $SU(n)$ is the explicit evaluation of the Clebsch-Gordan coefficient (CGC) occurring in the reduction of the direct product of two irreducible representations. A major difficulty in defining such coefficients arises out of specifying the product states unambiguously The group itself, except for $SU(2)$, does not provide enough labels, a fact known as the external labelling problem. Moshinsky (1963) has given a prescription for defining product states for the $SU(n)$ group and have, thereby, specified the general CGC From this definition Brody, Moshinsky & Renero (1965) have derived recursion formulas involving the auxiliary Wigner coefficients for the evalutation of the CGC Biedenharn *et al* (1967) have given a solution of the external labelling problem for $SU(3)$ with suggestions for extension to $SU(n)$.

The analytic expressions (Sharp *et al* 1967, Datta Majumdar *et al* 1973, Hecht 1965) for the CGC of $SU(3)$ as well as their explicit forms in some useful special cases (Kuriyan *et al* 1965, Mukunda *et al* 1965, de Swart 1965) are available in literature. In this work we propose a method of determining the recurrence relations between the CGC of $SU(n)$ involving different representation labels In fact, these relations are in terms of integrals of the type discussed by de Swart (1965) and used by us in a previous paper (Datta Majumdar *et al* 1973) in connection with the CGC of $SU(3)$. These integrals, however, act as a building block in the evaluation of the orthonormalised CGC Using the recurrence relations one can determine the expressions for CGC with arbitrary representation lebels from those involving simpler representations

2 General Recurrence Relations

For any $SU(n)$ the contiguous relations connecting the CGC with different representation labels can be derived by using the following properties

$$D^{\mu_1}_{\sigma_1,\sigma_1'}(\alpha)D^{\mu_2}_{\sigma_2,\sigma_2'}(\alpha) = \sum_{\mu,\sigma,\sigma',\gamma} \begin{pmatrix}\mu_1\,\mu_2\,\mu \\ \sigma_1\,\sigma_2\,\sigma\end{pmatrix}; \gamma\Big) \begin{pmatrix}\mu_1\,\mu_2\,\mu \\ \sigma'_1\sigma'_2\sigma'\end{pmatrix}; \gamma\Big) D^{\mu}_{\sigma,\sigma'}(\alpha) \quad \ldots \quad (1)$$

and

$$\int D^{\mu_1}_{\sigma_1,\sigma_1'}(\alpha)D^{\mu_2}_{\sigma_2,\sigma_2'}(\alpha)D^{\mu}_{\sigma,\sigma'}(\alpha)d\Omega = \frac{\int d\Omega}{N} \sum_{\gamma} \left(\begin{matrix}\mu_1\,\mu_2\,\mu \\ \sigma_1\,\sigma_2\,\sigma\end{matrix}; \gamma\right) \left(\begin{matrix}\mu_1\,\mu_2\,\mu \\ \sigma'_1\sigma'_2\sigma'\end{matrix}; \gamma\right) \quad \ldots \quad (2)$$

where $D^{\mu}_{\sigma,\sigma'}(\alpha)$ are the matrix elements of finite transformation, μ the representation labels, σ, σ' the state labels N the dimensionality of the representation μ, γ the degeneracy labels and $d\Omega$ the volume element in the group-space For actual calculations we start with the expression

$$I = \int D^{\mu *}_{\sigma,\sigma'}(\alpha)D^{\mu_1}_{\sigma_1,\sigma_2'}(\alpha)D^{\mu_2}_{\sigma_2,\sigma_2'}(\alpha)D^{\mu_0}_{\sigma_0,\sigma_0'}(\alpha)d\Omega. \quad \ldots \quad (3)$$

where μ_0 is one of the lower dimensional representations and σ_0, σ'_0 can be restricted conveniently Using eq (1) we combine $D^{\mu_0}_{\sigma_0,\sigma_0'}(\alpha)$ with one of the three matrix elements in turn and thus get three equivalent expressions for I. Equating any two of these expressions and using eq (2) we get contiguous relations between the C-G coefficients with different representation labels As an illustration we shall consider the groups $SU(2)$ and $SU(3)$.

3 Three Term Recurrence Relations for $SU(2)$

In the case of $SU(2)$ we start with the expression

$$I = \int D^{j}_{m,m'}(\alpha)D^{j_1}_{m_1,m_1'}(\alpha)D^{j_2}_{m_2,m_2'}(\alpha)D^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}(\alpha)d\Omega \quad \ldots \quad (4)$$

Following the above procedure we get the three equivalent expressions for I.

$$\frac{[(j_1+m_1+1)(j_1+m_1'+1)]^{\frac{1}{2}}}{(2j_1+1)(2j+1)} \begin{Bmatrix} j_1+\frac{1}{2} & j_2 & j \\ m_1+\frac{1}{2} & m_2 & m \end{Bmatrix} \begin{Bmatrix} j_1+\frac{1}{2} & j_2 & j \\ m_1'+\frac{1}{2} & m_2' & m' \end{Bmatrix}$$

$$+\frac{[(j_1-m_1)(j_1-m'_1)]^{\frac{1}{2}}}{(2j_1+1)(2j+1)} \begin{Bmatrix} j_1-\frac{1}{2} & j_2 & j \\ m_1+\frac{1}{2} & m_2 & m \end{Bmatrix} \begin{Bmatrix} j_1-\frac{1}{2} & j_2 & j \\ m_1'-\frac{1}{2} & m_2' & m' \end{Bmatrix}. \quad \ldots \quad (5a)$$

$=$ same expression with changes $j_1 \Longleftrightarrow j_2$, $m_1 \Longleftrightarrow m_2$, $m'_1 \Longleftrightarrow m'_2$ ... (5b)

$= (-)^{2j_2-m_2-m_2'} \times$ same expression with changes

$j \Longleftrightarrow j_1$, $m_1 \Longleftrightarrow m$, $m_1' \Longleftrightarrow m.'$ ... (5c)

Using the well known symmetries of the C–G coefficient of $SU(2)$ it can be shown that (5a) = (5b) goes over to (5a) = (5c) under the transformation $j_2 \Longleftrightarrow j$, $m_2 \Longleftrightarrow -m$, $m_2' \Longleftrightarrow -m'$ and to (5b) = (5c) under the transformation $j_1 \Longleftrightarrow j$, $m_1 \Longleftrightarrow -m$, $m'_1 \Longleftrightarrow -m'$. Hence only one of the three relations is independent. Substituting $m_1' = j_1$ in (5a) = (5b) we get the well known (Rose 1957, Edmonds 1957) three term recurrence relation

$$
\begin{aligned}
&(j_1+m_1+1)^{\frac{1}{2}} \begin{Bmatrix} j_1+\frac{1}{2} & j_2 & j \\ m_1+\frac{1}{2} & m_2 & m \end{Bmatrix} \\
&\quad = [(j_2+m_2+1)(j+j_1-j_2+\tfrac{1}{2})(j-j_1+j_2+\tfrac{1}{2})]^{\frac{1}{2}} \frac{1}{2j_2+1} \begin{Bmatrix} j_1 & j_2+\frac{1}{2} & j \\ m_1 & m_2+\frac{1}{2} & m \end{Bmatrix} \\
&\quad + \left[(j_2-m_2)(j_1+j_2-j+\tfrac{1}{2})\left(j_1+j_2+j+\frac{3}{2}\right)\right]^{\frac{1}{2}} \frac{1}{2j_2+1} \begin{Bmatrix} j_1 & j_2-\frac{1}{2} & j \\ m_1 & m_2-\frac{1}{2} & m \end{Bmatrix}
\end{aligned} \quad \ldots \quad (6)
$$

Other relations obtained by substituting $m'_1 = -j_1$, $m_2' = \pm j_2$ and $m' = \pm j$ can all be derived from eq (6) and its equivalent forms by using symmetries of the C–G coefficient of $SU(2)$. Further, the recurrence relations obtained by choosing $D^{\frac{1}{2}}_{\frac{1}{2},-\frac{1}{2}}(\alpha)$, $D^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\alpha)$ or $D^{\frac{1}{2}}_{-\frac{1}{2},-\frac{1}{2}}(\alpha)$ instead of $D^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\alpha)$ in eq (4) are also related to eq (6) through the symmetries of the C–G coefficient of $SU(2)$. Hence there is only one independent recurrence relation between the C–G coefficients of $SU(2)$.

4 Three Term Recurrence Relations for $SU(3)$

For $SU(3)$ we start with the expression

$$
I = \int D^{\mu *}_{\nu,\,\nu'}(\alpha) D^{\mu_1}_{\nu_1,\,\nu_1'}(\alpha) D^{\mu_2}_{\nu_2,\,\nu_2'}(\alpha) D^{\{3\}}_{\nu_0,\,\nu_0}(\alpha) d\Omega \quad \ldots \quad (7)
$$

where μ, ν, ν_0 and $\{3\}$ stands for (p, q), (j, m, γ), $(0, 0, -2/3)$ and $(1, 0)$ respectively. A graphical representation for such expressions with n-arbitrary matrix elements has been considered by El Baz *et al* (1967). Proceeding as above we get three equivalent expressions for I and hence three contiguous relations of which only one is independent. This we can write as

$$
\frac{[(p_1-j_1-\delta_1+1)(p_1+j_1-\delta_1+2)(p_1-j'_1-\delta'_1+1)(p_1+j_1'-\delta'_1+2)]^{\frac{1}{2}}}{(p_1+1)(p_1+q_1+2)}
$$

$$
\times \begin{pmatrix} \mu_{1a} & \mu_2 & \mu_\gamma \\ \bar{\nu}_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \mu_{1a} & \mu_2 & \mu_\gamma \\ \bar{\nu}_1' & \nu_2' & \nu' \end{pmatrix}
$$

$$+\frac{[(q_1-j_1+\delta_1)(q_1+j_1+\delta_1+1)(q_1-j_1'+\delta_1')(q_1+j_1'+\delta_1'+1)]^{\frac{1}{2}}}{(q_1+1)(p_1+q_1+2)}$$

$$\times\begin{pmatrix}\mu_{1b} & \mu_2 & \mu_\gamma\\ \bar{\nu}_1 & \nu_2 & \nu\end{pmatrix}\begin{pmatrix}\mu_{1b} & \mu_2 & \mu_\gamma\\ \bar{\nu}'_1 & \nu'_2 & \nu'\end{pmatrix}$$

$$+\frac{[(j_1+\delta_1)(j_1-\delta_1+1)(j_1+\delta_1')(j_1-\delta_1'+1)]^{\frac{1}{2}}}{(p_1+1)(q_1+1)}$$

$$\times\begin{pmatrix}\mu_{1c} & \mu_2 & \mu_\gamma\\ \bar{\nu}_1 & \nu_2 & \nu\end{pmatrix}\begin{pmatrix}\mu_{1c} & \mu_2 & \mu_\gamma\\ \bar{\nu}'_1 & \nu_2' & \nu'\end{pmatrix} \quad \ldots \quad (8)$$

$$=(-)^{2j_2-2j_1+2j_2'-2j_1'}\times\text{some expressions with changes } 1\Longleftrightarrow 2.$$

where $\bar{\nu}_1=\nu_1+\nu_0$, $\bar{\nu}'_1=\nu_1'+\nu_0$, $2\delta=Y-2/3(p-q)$ and μ_{1a}, μ_{1b}, μ_{1c} stands for (p_1+1, q_1), (p_1, q_1-1), (p_1-1, q_1+1) respectively. Similar relations obtained by using $D^{\{3\}}_{-\nu_0,\ \nu_0}(\alpha)$ in eq (7) are connected to eq (8) through the symmetries of C-G coefficient of $SU(3)$ and hence are redundant.

In addition to eq (8) we can get a twelve-term relation by using one of the matrix elements $D^{10}_{1/2\pm1/2\ 1/3},(\alpha)_{00-2/3}$, $D^{10}_{00-2/3},(\alpha)_{1/2\pm1/2\ 1/3}$ in the l h s. of eq (7) These four relations can be derived from one another by interchanging ν's with ν''s in some cases and through the combined use of the Wigner-Eckart theorem and symmetries of C–G coefficient of $SU(2)$ in the others Hence only one of them survives Using the remaining four matrix elements of the fundamental representation i e , $D^{10}_{\frac{1}{2}\pm\frac{1}{2}\ 1/3}$, $(\alpha)_{\frac{1}{2}\pm\frac{1}{2}\ 1/3}$ we obtain four equivalent 24-term relations between the products of C–G coefficients. However, as we have to restrict one of the C–G coefficients in the product to get a linear recurrsion formula, these 24-term relations yield the same result as the twelve-term relations Hence for $SU(3)$ there can be almost two independent recurrence relations between the C–G coefficients involving different μ's

Owing to the summation over γ in eq. (8) it is very difficult to find out a linear relation between the orthogonal C–G coefficients In fact eq (8) gives us a connection between the integrals of the type occurring in the l h s. of eq (2) These integrals, however, form a foundation stone for determining the orthogonalised C–G coefficient In the non-degenerate case the ratio $I(\nu_1\nu'_1, \nu_2\nu'_2, \nu\nu')/I(\nu'_1\nu'_1, \nu'_2\nu'_2, \nu'\nu')^{\frac{1}{2}}$ is the C-G coefficient with state labels ν_1, ν_2, ν In the degenerate case each integral with same ν's but different ν''s can be treated as a non-orthogonal C--G coefficient. These can be orthogonalised by using Schmidt's procedure where the coefficients involve the above integrals only. Hence the orthogonalised C–G coefficient can be determined once the integrals are known.

Thus eq (8) gives us a method of building up the C–G coefficient containing higher representations from those containing the fundamental representation and its conjugate. Making one of the integers p_1, q_1, p_2 or q_2 equal to zero we get a connection between the integrals in the degenerate case and those in the non-degenerate case.

In the non-degenerate case a linear recurrence relation between the isoscalar factors can be obtained by suitably restricting ν''s (or ν's) in eq (7). However for $SU(3)$ it is not possible to get a set of values for ν''s which gives a simple closed form for the C–G coefficient without restricting μ as well. Choosing $j'_1 = j'_2 = j' = 0$ and $q_2 = 0$ in eq. (7), we get a three-term relation

$$\left[\frac{(p_1-j_1-\delta_1+1)(p_1+j_1-\delta_1+2)(p_2+1-\lambda)(p_1+q_1+2-\lambda)}{(p_1+1)}\right]^{\frac12}\begin{pmatrix} p_1+1q_1 & p_20 & pq \\ j_1\delta_1 & j_2j_2 & j\delta \end{pmatrix}$$

$$+\left[\frac{(q_1-j_1+\delta_1)(q_1+j_1+\delta_1+1)\lambda(p+q+2+\lambda)}{(q_1+1)}\right]^{\frac12}\begin{pmatrix} p_1q_1-1 & p_20 & pq \\ j_1\delta_1 & j_2j_2 & j\delta \end{pmatrix}$$

$$=(-)^{2j_2-2j_1}[(p_1+q_1+2)(p_2-2j+1)]^{\frac12}\begin{pmatrix} p_1q_1 & p_2+1\ 0 & pq \\ j_1\delta_1 & j_2j_2 & j\delta \end{pmatrix} \quad \ldots \quad (9)$$

where $\lambda = p_1+p_2+1-p = q_1-q$ is a non-negative integer

Similar relations for other non-degenerate cases where one of the integers p_1 q_1 or p_2 is zero can be derived from eq (9) by using the symmetries of the C–G coefficient of $SU(3)$

Another three-term relation between the isocalar factors can be obtained by starting with the expression

$$I = \int D^{p_1q_1}_{j_1m_1\delta_1'}(\alpha)_{\frac12 p_1 m_1' \frac12 p_1'} D^{p_2q_2}_{j_2m_2\delta_2}(\alpha)_{\frac12 p_2 \frac12 p_2 \frac12 p_2'} D^{pq}_{jm\delta}(\alpha)_{\frac12 pm' \frac12 p} D^{10}{}_{000'}(\alpha)_{\frac12\frac12\frac12} d\Omega$$

Combining $D^{10}{}_{000'}(\alpha)_{\frac12\frac12\frac12}$ with the first and second matrix elements alternately we get the relation

$$\left[\frac{(p_1-j_1-\delta_1+1)(p_1+j_1-\delta_1+2)(p_2+1-\lambda)(p_1+1-\lambda)}{(p_1+q_1+2)}\right]^{\frac12}\begin{pmatrix} p_1+1\ q_1 & p_2q_2 & pq \\ j_1\delta_1 & j_2\delta_2 & j\delta \end{pmatrix}$$

$$-\left[\frac{(j_1+\delta_1)(j_1-\delta_1+1)\lambda(p_1+p_2+2-\lambda)}{(q_1+1)}\right]^{\frac12}\begin{pmatrix} p_1-1\ q_1+1 & p_2q_2 & pq \\ j_1\delta_1-1 & j_2\delta_2 & j\delta \end{pmatrix}$$

$$=(-)^{2j_2-2j_1}(p_1+1)\left[\frac{(p_2-j_2-\delta_2+1)(p_2+j_2-\delta_2+2)}{(p_2+q_2+2)}\right]^{\frac12}\begin{pmatrix} p_1q_1 & p_2+1\ q_2 & pq \\ j_1\delta_1 & j_2\delta_2 & j\delta \end{pmatrix}$$

$$\ldots \quad (10)$$

where $(p\ q)$ is the non-degenerate product representation belonging to the class $p = p_1 + p_2 + 1 - 2\lambda$, $q = q_1 + q_2 + \lambda$ with λ a non-negative integer The range of p, q values for which the above relation is valid can be further extended by using the symmetry of the C–G coefficient

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